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STOCHASTIC DOMINANCE FOR SEQUENCES AND IMPLIED UTILITY IN DYNAMIC OPTIMIZATION

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Abstract

We introduce a stochastic dynamic optimization problem, where risk aversion is expressed by a stochastic ordering constraint. The constraint requires that a random reward sequence depending on our decisions dominates a given benchmark random sequence. The dominance is defined by discounting both processes with a family of discount sequences, and by applying a univariate order. We describe the generator of this order. We develop necessary and sufficient conditions of optimality for convex stochastic control problems with the new ordering constraint and we derive an equivalent control problem featuring implied utility functions.

Key words: stochastic control, stochastic programming, stochastic orders, risk, utility

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1. Introduction. In our earlier publications [2–5] we have introduced and analysed the following optimization model with stochastic dominance constraints:

$$\max \mathbb{E}\big[H(z)\big] \tag{1}$$

$$\text{s.t.}\ \ G(z) \succeq_{(2)} Y, \tag{2}$$

$$z \in Z_0. \tag{3}$$

In this problem Z_0 is a convex closed subset of a Banach space $\mathcal{Z}$, and G and H are continuous operators from $\mathcal{Z}$ to the space of integrable random variables

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$\mathcal{L}_1(\Omega, \mathcal{F}, P)$. The random variable Y plays the role of a benchmark outcome. The relation $\succeq_{(2)}$ used in (2) is the stochastic dominance relation of the second order (see [8,9] and the references therein).

Our objective is to extend model (1)–(3) to a dynamic setting, with $G(z)$ representing a random sequence, rather than a scalar random variable. We are interested in modelling risk aversion in a stochastic control problem for a discrete-time linear dynamic system governed by the equations

$$s_{t+1} = A_t s_t + B_t v_t + e_t, \quad t = 1, \dots, T.$$

Here s_t denotes the state vector at time t and v_t denotes the control vector. The vectors e_t and the matrices A_t and B_t are random. The initial state s_1 is given.

To formulate the problem, assume that on the probability space $(\Omega, \mathcal{F}, P)$ we have a filtration $\mathcal{F}_1 \subset \mathcal{F}_2 \cdots \subset \mathcal{F}_{T+1}$, with $\mathcal{F}_1 = \{\varnothing, \Omega\}$ and $\mathcal{F}_{T+1} = \mathcal{F}$. The σ-field $\mathcal{F}_t$ is generated by the information available at time t, when control v_t is chosen. We assume that $e_t \in \mathcal{L}_p^{n_s}(\Omega, \mathcal{F}_{t+1}, P)$, $v_t \in \mathcal{L}_p^{n_v}(\Omega, \mathcal{F}_t, P)$, $s_t \in \mathcal{L}_p^{n_s}(\Omega, \mathcal{F}_t, P)$, with some $p \in [1, \infty)$. The matrices A_t and B_t are elements of spaces of matrices of appropriate dimensions which are measurable with respect to $\mathcal{F}_t$ and essentially bounded. The standard symbol $\mathcal{L}_p^m(\Omega, \mathcal{F}, P)$ denotes the space of all $\mathcal{F}$-integrable mappings $X : \Omega \to \mathbb{R}^m$, for which $\mathbb{E}\|X\|^p < \infty$. If the values are taken in $\mathbb{R}$, the superscript m is omitted.

Specific conditions impose additional constraints on our actions: $v_t \in V_t$, P-a.s, where each V_t is a convex closed set in $\mathbb{R}^{n_v}$.

Assume that the random outcomes X_t representing the performance measures of the system at $t = 1, \dots, T+1$, are scalar and given by

$$\begin{aligned} X_t(\omega) &= g_t(s_t(\omega), v_t(\omega)), \quad \text{for } t = 1, \dots, T, \\ X_{T+1}(\omega) &= g_{T+1}(s_{T+1}(\omega)), \quad \omega \in \Omega. \end{aligned} \tag{4}$$

The functions $g_t : \mathbb{R}^{n_s} \times \mathbb{R}^{n_v} \to \mathbb{R}$ are concave and satisfy the growth condition: $|g_t(s, v)| \le C_1 + C_2\big(\|s\|^p + \|v\|^p\big)$, with some constants C_1 and C_2. If $p = 1$ this condition (for concave functions) amounts to global Lipschitz continuity.

We adopt the convention that larger values of X_t are preferred.

Relations (4) define mappings $G_t : \mathcal{L}_p^{n_s}(\Omega, \mathcal{F}_t, P) \times \mathcal{L}_p^{n_v}(\Omega, \mathcal{F}_t, P) \to \mathcal{L}_1(\Omega, \mathcal{F}_t, P)$, $t = 1, \dots, T$, and $G_{T+1} : \mathcal{L}_p^{n_s}(\Omega, \mathcal{F}_{T+1}, P) \to \mathcal{L}_1(\Omega, \mathcal{F}_{T+1}, P)$, in the following way:

$$\begin{aligned} [G_t(s_t, v_t)](\omega) &= g_t(s_t(\omega), v_t(\omega)), \quad t = 1, \dots, T, \\ [G_{T+1}(s_{T+1})](\omega) &= g_{T+1}(s_{T+1}(\omega)), \quad \omega \in \Omega. \end{aligned}$$

We write $G(s, v) = \big(G_1(s_1, v_1), \dots, G_T(s_T, v_T), G_{T+1}(s_{T+1})\big)$. Here $s = (s_1, \dots, s_{T+1})$ and $v = (v_1, \dots, v_T)$ represent the variables of our problem.

Our goal is to model risk aversion in dynamic problem by using stochastic orders. To this end we compare the multivariate distribution of the rewards

$(X_1, X_2, \dots, X_{T+1})$ with the distribution of some benchmark outcomes $(Y_1, Y_2, \dots, Y_{T+1})$.

The space of continuous functions on a compact set $D \subset \mathbb{R}^n$ is denoted $\mathcal{C}(D)$. The space of regular countably additive measures on a compact set $D \subset \mathbb{R}^n$ having finite variation is denoted $\mathcal{M}(D)$; its subset of nonnegative measures is denoted by $\mathcal{M}_+(D)$. For a Banach space $\mathcal{Z}$ we denote its topological dual by $\mathcal{Z}^*$. We denote the space of the random outcomes by $\mathcal{X} = \mathcal{L}_1(\Omega, \mathcal{F}_1, P) \times \dots \times \mathcal{L}_1(\Omega, \mathcal{F}_{T+1}, P)$. Its dual is $\mathcal{X}^* = \mathcal{L}_\infty(\Omega, \mathcal{F}_1, P) \times \dots \times \mathcal{L}_\infty(\Omega, \mathcal{F}_{T+1}, P)$.

2. Stochastic dominance for random reward sequences. The notion of stochastic ordering for scalar random variables (or stochastic dominance of first order) is defined as follows. For a random variable X we consider its distribution function, $F(X;\eta) = P[X \le \eta]$, $\eta \in \mathbb{R}$. We say that a random variable X dominates in the first order a random variable Y if $F(X;\eta) \le F(Y;\eta) \quad \forall \quad \eta \in \mathbb{R}$. We denote this relation $X \succeq_{(1)} Y$.

Consider a scalar random variable $X \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$ and define $F_2(X;\eta) = \int_{-\infty}^{\eta} F(X;\alpha)\; d\alpha$, for $\eta \in \mathbb{R}$. A random variable $X \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$ dominates in the second order a random variable $Y \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$ if $F_2(X;\eta) \le F_2(Y;\eta)$ for all $\eta \in \mathbb{R}$. We denote this relation as $X \succeq_{(2)} Y$. Equivalently: $\mathbb{E}[(\eta - X)_+] \le \mathbb{E}[(\eta - Y)_+]$ for all $\eta \in \mathbb{R}$.

Let us consider the set $\mathcal{U}$ of concave nondecreasing functions $u : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions: $\lim\limits_{t \to -\infty} u(t)/t < \infty$, and $\lim\limits_{t \to \infty} u(t) = 0$. With every $u \in \mathcal{U}$ we can associate a measure ν on $\mathbb{R}$ as follows: $\nu([\tau, \infty)) = u'_-(\tau)$, where u'_- is the left derivative of u. Representing $u(t) = -\int_t^\infty u'_-(\tau)\; d\tau$ and integrating by parts, we get

$$u(t) = -\int_{-\infty}^{\infty} \max(0, \eta - t)\; \nu(d\eta). \tag{5}$$

It is is well-known that the relation $X \succeq_{(2)} Y$ is equivalent to $\mathbb{E}[u(X)] \ge \mathbb{E}[u(Y)]$ for all $u \in \mathcal{U}$.

Consider random vectors $(X_1, \dots, X_{T+1})$ and $(Y_1, \dots, Y_{T+1})$ in $\mathcal{L}_1^{T+1}(\Omega, \mathcal{F}, P)$. Let

$$\mathcal{D} \subseteq \big\{\rho \in \mathbb{R}^{T+1} : 1 \ge \rho_1 \ge \rho_2 \ge \dots \ge \rho_{T+1} \ge 0\big\}.$$

Definition 1. A random sequence $(X_1, \dots, X_{T+1}) \in \mathcal{X}$ dominates a random sequence
$(Y_1, \dots, Y_{T+1}) \in \mathcal{X}$ in the discounted second order, if for all $\rho \in \mathcal{D}$ the relation $\sum_{t=1}^{T+1} \rho_t X_t \succeq_{(2)} \sum_{t=1}^{T+1} \rho_t Y_t$ is satisfied.

We denote this relation by $X \succeq_{(2)}^{\text{dis}} Y$. For brevity we write $\langle \rho, X \rangle = \sum_{t=1}^{T+1} \rho_t X_t$. We obtain

$$\mathbb{E}\big[u(\langle \rho, X \rangle)\big] \ge \mathbb{E}\big[u(\langle \rho, Y \rangle)\big] \quad \text{for all } u \in \mathcal{U} \text{ and all } \rho \in \mathcal{D}. \tag{6}$$

Using (5), we conclude that for every nonnegative measure ν on $\mathbb{R}$ and for every $\rho \in \mathcal{D}$

$$\int\limits_{\Omega}\int\limits_{\mathbb{R}} \max(0,\eta-\langle\rho,X(\omega)\rangle)\ \nu(d\eta)\ P(d\omega) \le \int\limits_{\Omega}\int\limits_{\mathbb{R}} \max(0,\eta-\langle\rho,Y(\omega)\rangle)\ \nu(d\eta)\ P(d\omega).$$

For every $\lambda \in \mathcal{M}_+(\mathbb{R}\times\mathcal{D})$ we define a concave nondecreasing function $\varphi_\lambda : \mathbb{R}^{T+1} \to \mathbb{R}$ as

$$\varphi_\lambda(x) = -\int\limits_{\mathbb{R}\times\mathcal{D}} \max(0,\eta-\langle\rho,x\rangle)\ \lambda(d(\eta,\rho)).$$

Proposition 1. For each $X, Y \in \mathcal{X}$ the relation $X \succeq_{(2)}^{\rm dis} Y$ is equivalent to

$$\mathbb{E}[\varphi_\lambda(X)] \ge \mathbb{E}[\varphi_\lambda(Y)] \quad \text{for all } \lambda \in \mathcal{M}_+(\mathbb{R}\times\mathcal{D})\}. \tag{7}$$

Proof. Assume $X \succeq_{(2)}^{\rm dis} Y$. Take any $\lambda \in \mathcal{M}_+(\mathbb{R}\times\mathcal{D})$. We shall show inequality (7) for φ_λ. Let λ_ρ be the conditional measure of λ on $\mathbb{R}$, for $\rho \in \mathcal{D}$ (see, e.g., [7], Theorem 10.2.2). Denote by μ the marginal measure of λ on $\mathcal{D}$. The integral over $\mathbb{R}\times\mathcal{D}$ can be written as an iterated integral (see, e.g., [7], Theorem 10.21):

$$\varphi_\lambda(x) = -\int\limits_{\mathcal{D}}\int\limits_{\mathbb{R}} \max(0,\eta-\langle\rho,x\rangle)\ \lambda_\rho(d\eta)\ \mu(d\rho).$$

From the definition of $\succeq_{(2)}^{\rm dis}$ and (6) it follows that

$$\int\limits_{\Omega}\int\limits_{\mathbb{R}} \max(0,\eta-\langle\rho,X(\omega)\rangle)\ \lambda_\rho(d\eta)\ P(d\omega) \le \int\limits_{\Omega}\int\limits_{\mathbb{R}} \max(0,\eta-\langle\rho,Y(\omega)\rangle)\ \lambda_\rho(d\eta)\ P(d\omega).$$

Integrating with the measure μ and changing the order of integration, we obtain

$$\begin{aligned}\mathbb{E}[\varphi_\lambda(X)] &= -\int\limits_{\Omega}\int\limits_{\mathcal{D}}\int\limits_{\mathbb{R}} \max(0,\eta-\langle\rho,X(\omega)\rangle)\ \lambda_\rho(d\eta)\ \mu(d\rho)\ P(d\omega)\\ &\ge -\int\limits_{\Omega}\int\limits_{\mathcal{D}}\int\limits_{\mathbb{R}} \max(0,\eta-\langle\rho,Y(\omega)\rangle)\ \lambda_\rho(d\eta)\ \mu(d\rho)\ P(d\omega) = \mathbb{E}[\varphi_\lambda(Y)],\end{aligned}$$

as required. To prove the converse, we observe that we can choose measures λ such that their marginal measures μ on $\mathcal{D}$ are atomic. The last displayed inequality becomes equivalent to the inequality in (6). This is equivalent to the definition of the order $\succeq_{(2)}^{\rm dis}$. □

Another way to characterize the relation $\succeq_{(2)}^{\rm dis}$ is to use the integrated distribution functions:

$$F_2(\langle\rho,X\rangle;\eta) \le F_2(\langle\rho,Y\rangle;\eta) \quad \text{for all } \rho \in \mathcal{D} \text{ and all } \eta \in \mathbb{R}. \tag{8}$$

3. Optimality conditions. The implied utility functions. We introduce the stochastic dynamic optimization problem with discounted dominance constraints

$$(9)\qquad \begin{aligned} &\max \sum_{t=1}^{T} \mathbb{E}G_t(s_t, v_t) + \mathbb{E}G_{T+1}(s_{T+1}) \\ &\text{s.t.}\ \ s_{t+1} = A_t s_t + B_t v_t + e_t, \quad t = 1, \dots, T, \\ &\qquad (G_1(s_1, v_1), \dots, G_1(s_T, v_T), G_{T+1}(s_{T+1})) \succeq^{\text{dis}}_{(2)} (Y_1, \dots, Y_T, Y_{T+1}) \\ &\qquad v_t \in V_t \text{ a.s.}, \quad t = 1, \dots, T. \end{aligned}$$

Define the space of controls $(v_1, \dots, v_T)$ by $\mathcal{V} = \mathcal{L}_p^{n_v}(\Omega, \mathcal{F}_1, P) \times \cdots \times \mathcal{L}_p^{n_v}(\Omega, \mathcal{F}_T, P)$. The space of state trajectories $(s_2, \dots, s_{T+1})$ is denoted by $\mathcal{S} = \mathcal{L}_p^{n_s}(\Omega, \mathcal{F}_2, P) \times \cdots \times \mathcal{L}_p^{n_s}(\Omega, \mathcal{F}_{T+1}, P)$.

The relation $X \succeq^{\text{dis}}_{(2)} Y$ can be equivalently formulated as (8). For technical reasons, which will become apparent later, we restrict the range of $\eta \in \mathbb{R}$, for which we impose (8) to an interval $[a, b]$, and transform problem (9) to the following:

$$(10)\qquad \max \sum_{t=1}^{T} \mathbb{E}G_t(s_t, v_t) + \mathbb{E}G_{T+1}(s_{T+1})$$

$$(11)\qquad \text{s.t.}\ \ s_{t+1} = A_t s_t + B_t v_t + e_t, \quad t = 1, \dots, T,$$

$$(12)\qquad F_2(\langle \rho, G(s, v) \rangle; \eta) \le F_2(\langle \rho, Y \rangle; \eta) \text{ for all } \rho \in \mathcal{D} \text{ and all } \eta \in [a, b],$$

$$(13)\qquad v_t \in V_t \text{ a.s.}, \quad t = 1, \dots, T.$$

Define the set $\mathcal{U}([a, b])$ of functions $u(\cdot)$ satisfying the following conditions: $u(\cdot)$ is concave and nondecreasing; $u(t) = 0$ for all $t \ge b$; and $u(t) = u(a) + \gamma(t - a)$, with $\gamma > 0$, for all $t \le a$.

Define the class of functions

$$\Phi([a, b], \mathcal{D}) = \Big\{ -\int_{[a,b]\times\mathcal{D}} \max(0, \eta - \langle \rho, x \rangle)\, \lambda(d(\eta, \rho)) :\ \lambda \in \mathcal{M}_+([a, b] \times \mathcal{D}) \Big\}.$$

By the same argument as Proposition 1, the set $\Phi([a, b], \mathcal{D})$ is a generator of the order (12).

We introduce the functional $L : \mathcal{S} \times \mathcal{V} \times \Phi([a, b], \mathcal{D}) \to \mathbb{R}$, which plays the role of a partial Lagrangian associated with problem (10)-(13):

$$\begin{aligned} L(s, v, \varphi) = \mathbb{E}\Big[&\sum_{t=1}^{T} G_t(s_t, v_t) + G_{T+1}(s_{T+1}) \\ &+ \Big(\varphi(G_1(s_1, v_1), \dots, G_T(s_T, v_T), G_{T+1}(s_{T+1})) - \varphi(Y_1, \dots, Y_T, Y_{T+1}) \Big) \Big]. \end{aligned}$$

For a fixed $\varphi(\cdot)$, we use L as an objective functional in an auxiliary control problem

$$\text{(14)}\qquad \max\ \mathbb{E}\Big[\sum_{t=1}^{T} G_t(s_t,v_t) + G_{T+1}(s_{T+1}) + \Big(\varphi(G_1(s_1,v_1),\dots,G_T(s_T,v_T),G_{T+1}(s_{T+1})) - \varphi(Y_1,\dots,Y_T,Y_{T+1})\Big)\Big]$$

$$\text{(15)}\qquad \text{s.t. } s_{t+1} = A_t s_t + B_t v_t + e_t, \quad t=1,\dots,T,$$

$$\text{(16)}\qquad v_t \in V_t \text{ a.s.}, \quad t=1,\dots,T.$$

Let Z_0 denote the convex set of (s,v) satisfying conditions (15)–(16). The following property plays the role of a constraint qualification condition.

Definition 2. Problem (10)–(13) satisfies the uniform dominance condition if there exists a pair $(\tilde{s},\tilde{v}) \in Z_0$ such that

$$\inf_{(\eta,\rho)\in[a,b]\times\mathcal{D}} \Big\{ F_2(\langle\rho,Y\rangle;\eta) - F_2(\langle\rho,G(\tilde{s},\tilde{v})\rangle;\eta)\Big\} > 0.$$

This condition is the reason for considering constraints (11) in the finite interval $[a,b]$. A constraint qualification of Slater type cannot be satisfied for $\eta \in \mathbb{R}$ since for any random variable X the function $F_2(X;\eta)$ converges to zero, whenever $\eta \to -\infty$.

Theorem 1. Assume that the uniform dominance condition is satisfied. If $(\hat{s},\hat{v})$ is an optimal solution of (10)–(13) then there exist $\hat{\varphi} \in \Phi([a,b],\mathcal{D})$ such that $(\hat{s},\hat{v})$ is an optimal solution of problem (14)–(16) with $\varphi = \hat{\varphi}$, and

$$\text{(17)}\qquad \mathbb{E}[\hat{\varphi}(G(\hat{s},\hat{v}))] = \mathbb{E}[\hat{\varphi}(Y)].$$

Conversely, if for some function $\hat{\varphi} \in \Phi([a,b],\mathcal{D})$ an optimal solution $(\hat{s},\hat{v})$ of (14)–(16) satisfies (12) and (17), then $(\hat{s},\hat{v})$ is an optimal solution of (10)–(13).

Proof. Let us rewrite (12) as $\Gamma(s,v) \in K$, where $\Gamma : \mathcal{S}\times\mathcal{V} \to \mathcal{C}([a,b]\times\mathcal{D})$ is a continuous operator defined as

$$[\Gamma(s,v)](\eta,\rho) = F_2(\langle\rho,Y\rangle;\eta) - F_2(\langle\rho,G(s,v)\rangle;\eta), \quad \eta\in[a,b],\ \rho\in\mathcal{D}.$$

The set K is the cone of nonnegative functions in $\mathcal{C}([a,b]\times\mathcal{D})$. Observe that for every $\rho \in \mathcal{D}$ the function $(s,v) \to \eta - \langle\rho,G(s,v)\rangle$ is convex, for almost all $\omega \in \Omega$, and the function $x \to (x)_+$ is convex and nondecreasing. Therefore, the composition $F_2(\langle\rho,G(s,v)\rangle;\eta) = \mathbb{E}[(\eta - \langle\rho,G(s,v)\rangle)_+]$ is a convex function of (s,v). It follows that the operator Γ is concave with respect to the cone K. The dual space to $\mathcal{C}([a,b]\times\mathcal{D})$ is the space $\mathcal{M}([a,b]\times\mathcal{D})$. We introduce the Lagrangian $\Lambda : \mathcal{S}\times\mathcal{V}\times\mathcal{M}_+([a,b]\times\mathcal{D}) \to \mathbb{R}$,

$$\text{(18)}\qquad \Lambda(s,v,\lambda) = \sum_{t=1}^{T}\mathbb{E}G_t(s_t,v_t) + \mathbb{E}G_{T+1}(s_{T+1}) + \int_{[a,b]\times\mathcal{D}} [\Gamma(s,v)](\eta,\rho)\ \lambda(d(\eta,\rho)).$$

The uniform dominance condition implies that the following generalized Slater condition is satisfied: There exists a point $(\tilde{s}, \tilde{v}) \in Z_0$ such that $\Gamma(\tilde{s}, \tilde{v}) \in \operatorname{int} K$. Moreover, $(\tilde{s}, \tilde{v}) \in Z_0$. By ([1] Prop. 2.106) this is equivalent to the regularity condition: $0 \in \operatorname{int}[\Gamma(Z_0) - K]$. Therefore, we can use the necessary conditions of optimality in Banach spaces (see, e.g., [1], Theorem 3.4). We conclude that there exists a measure $\hat{\lambda} \in \mathcal{M}_+([a,b] \times \mathcal{D})$ such that

$$\Lambda(\hat{s}, \hat{v}, \hat{\lambda}) = \max_{(s,v)\in Z_0} \Lambda(s, v, \hat{\lambda}) \tag{19}$$

and

$$\int_{[a,b]\times\mathcal{D}} [F_2(\langle\rho, Y\rangle; \eta) - F_2(\langle\rho, G(\hat{s}, \hat{v})\rangle; \eta)]\ \hat{\lambda}(d(\eta, \rho)) = 0. \tag{20}$$

We shall transform these conditions to the postulated form. Using the representation of F_2 as expected shortfall and changing the order of integration, we obtain

$$\int_{[a,b]\times\mathcal{D}} F_2(\langle\rho, Y\rangle; \eta)\ \hat{\lambda}(d(\eta, \rho)) =$$

$$\int_{[a,b]\times\mathcal{D}} \int_{\Omega} \max(0, \eta - \langle\rho, Y(\omega)\rangle)\ P(d\omega)\ \hat{\lambda}(d(\eta, \rho)) = -\mathbb{E}\hat{\varphi}(Y).$$

with

$$\hat{\varphi}(x) = -\int_{[a,b]\times\mathcal{D}} \max(0, \eta - \langle\rho, x\rangle)\ \hat{\lambda}(d(\eta, \rho)), \qquad \hat{\varphi} \in \Phi([a,b] \times \mathcal{D}).$$

Thus, condition (19) implies the optimality in problem (14)–(16), and condition (20) implies (17).

Let us now prove the converse. If $\hat{\varphi} \in \Phi([a,b] \times \mathcal{D})$, then there exists a measure $\hat{\lambda} \in \mathcal{M}_+([a,b] \times \mathcal{D})$ such that $\hat{\varphi}(x) = -\int_{[a,b]\times\mathcal{D}} \max(0, \eta - \langle\rho, x\rangle)\ \lambda(d(\eta, \rho))$ and therefore $\mathbb{E}\hat{\varphi}(Y) = -\int_{[a,b]\times\mathcal{D}} F_2(\langle\rho, Y\rangle; \eta)\ \hat{\lambda}(d(\eta, \rho))$. Thus, the maximizer $(\hat{s}, \hat{v})$ of problem (14)–(16) is also the maximizer of $\Lambda(s, v, \hat{\lambda})$. It follows from sufficient conditions of optimality (see, e.g., [1] Prop. 3.3) that if $(\hat{s}, \hat{v})$ satisfies (11) and (17) then it is optimal for (10)–(13). □

Theorem 1 can be used to derive more specific optimality conditions, in particular, a maximum principle.

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